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On the rational recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-m}}$ for small A

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Abstract

This work studies the existence of positive prime periodic solutions of higher order for rational recursive equations of the form

$$y_n = A + \frac{y_{n-1}}{y_{n-m}}, \quad n = 0, 1, 2, \ldots,$$

with $y_{-m}, y_{-m+1}, \ldots, y_{-1} \in (0, \infty)$ and $m \in \{2, 3, 4, \ldots\}$. In particular, we show that for sufficiently small $A > 0$, there exist periodic solutions with prime period $2m + U_m + 1$, for almost all m, where $U_m = \max\{i \in \mathbb{N} : i(i + 1) \leq 2(m - 1)\}$.

Keywords: Rational difference equation; Periodicity; Existence; Binomial coefficients; Fixed point

1. Introduction

This note studies the existence of prime periodic solutions of higher order for rational recursive equations of the form

$$y_n = A + \frac{y_{n-1}}{y_{n-m}}, \quad n = 0, 1, \ldots, \tag{1}$$

with $A > 0$, $y_{-m}, y_{-m+1}, \ldots, y_{-1} \in (0, \infty)$ and $m \in \{2, 3, 4, \ldots\}$. Eq. (1) has been studied by many authors in the recent past. In [1], conditions for global asymptotic stability of solutions are presented. In [2], some quantitative bounds for solutions are provided. Properties of solutions for $A < 0$ are considered in [3,4]. Results for instances of the more general equation

$$y_n = A + \frac{y_{n-k}}{y_{n-m}}, \quad n = 0, 1, \ldots, \tag{2}$$

$k, m \in \{1, 2, 3, \ldots\}$, can be found in [5–17] and the references therein.

It is known (cf. [1]) that all positive solutions to (1) are bounded and persist, and that a sufficient condition for global asymptotic stability of the positive equilibrium of Eq. (1) is $A > 1$, but little is known regarding the possible

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behavior of solutions for small $A > 0$ and large m. Here we will show that for almost all m, for sufficiently small A, there exists a prime period $2m + U_m + 1$ solution to (1), where $U_m = \max\{i \in \mathbb{N} : i(i+1) \leq 2(m-1)\}$. In particular, we will prove the following theorem.

Theorem 1. *Set* $\mathcal{V} = \bigcup_{j>0} \left\{ \frac{j(j+1)}{2}, \frac{j(j+1)}{2} + 1 \right\}$. *If* $m > 1$ *satisfies* $m \notin \mathcal{V}$ *then there exists an* $\epsilon_m > 0$ *such that for all* $0 < A < \epsilon_m$, *there exists a prime period* $2m + U_m + 1$ *solution to* (1).

Proof. First, suppose that

$$y_i \in [A, A + 2A^2] \tag{3}$$

for $-m \leq i \leq -1$. We will show first that there exist $\{l_0, l_1, \ldots, l_{m-1}\}$ and $\{r_0, r_1, \ldots, r_{m-1}\}$ such that

$$y_i \in [A^{-i}(1 - l_i A), A^{-i}(1 + r_i A)], \tag{4}$$

for $0 \leq i \leq m - 1$. Indeed, note that for initial values satisfying (3),

$$y_0 \leq A + \frac{A + 2A^2}{A} = 1 + 3A \tag{5}$$

and

$$y_0 \geq A + \frac{A}{A + 2A^2} = A + \frac{1}{1 + 2A} \geq A + (1 - 2A) = 1 - A, \tag{6}$$

for $A > 0$ sufficiently small. Hence, employing (5) and (6) gives

$$y_1 \leq A + \frac{1 + 3A}{A} = A^{-1} + 3 + A \leq A^{-1}(1 + 4A) \tag{7}$$

and

$$y_1 \geq A + \frac{1 - A}{A(1 + 2A)} \geq A^{-1}(1 - A)(1 - 2A) \geq A^{-1}(1 - 3A), \tag{8}$$

for $A > 0$ sufficiently small. For convenience, throughout this proof, at each instance, $A > 0$ is assumed to be sufficiently small so that the associated inequality holds.

More generally, suppose

$$A^{-j}(1 - l_j A) \leq y_j \leq A^{-j}(1 + r_j A), \tag{9}$$

for $0 \leq j < J$ for some $2 \leq J \leq m - 1$, where $l_j, r_j > 0$, for $0 \leq j < J$. Then,

$$y_J \leq A + \frac{A^{-(J-1)}(1 + r_{J-1}A)}{A} \leq A^{-J}(1 + (r_{J-1} + 1)A) \tag{10}$$

and

$$y_J \geq A + \frac{A^{-(J-1)}(1 - l_{J-1}A)}{A(1 + 2A)} \geq A^{-J}(1 - (l_{J-1} + 2)A). \tag{11}$$

Setting $r_J = r_{J-1} + 1$ and $l_J = l_{J-1} + 2$, the inequalities in (4) then follow from induction.

Now, consider y_j for $m \leq j \leq m + U_m$, where $U_m = \max\{i \in \mathbb{N} : i(i+1) \leq 2(m-1)\}$. Employing the bounds in (4) gives (for sufficiently small $A > 0$) that

$$\begin{aligned}
y_m &\leq A + \frac{A^{-(m-1)}(1 + r_{m-1}A)}{(1 - l_0 A)} = A + A^{-(m-1)}(1 + r_{m-1}A)\left(1 + l_0 A + \left(\frac{(l_0^2 A)}{1 - l_0 A}\right) A\right) \\
&\leq A + A^{-(m-1)}(1 + r_{m-1}A)(1 + (l_0 + 1)A) \\
&\leq A^{-(m-1)}(1 + (r_{m-1} + l_0 + 1 + A^{m-1} + r_{m-1}(l_0 + 1)A)A) \\
&\leq A^{-(m-1)}(1 + r_m A),
\end{aligned} \tag{12}$$

where $r_m = r_{m-1} + l_0 + 3$. Similarly, we have

$$\begin{aligned} y_m &\geq A + \frac{A^{-(m-1)}(1 - l_{m-1}A)}{(1 + r_0 A)} = A + A^{-(m-1)}(1 - l_{m-1}A)\left(1 - r_0 A + \frac{(r_0 A)^2}{1 - l_0 A}\right) \\ &\geq A^{-(m-1)}(1 - l_{m-1}A)(1 - r_0 A) \geq A^{-(m-1)}(1 - (l_{m-1} + r_0)A) \\ &= A^{-(m-1)}(1 - l_m A), \end{aligned} \tag{13}$$

where $l_m = l_{m-1} + r_0$.

Inductively, employing (4), we obtain

$$y_i \in (A^{v_i}(1 - l_i A), A^{v_i}(1 + r_i A)) \tag{14}$$

for $m \leq i \leq m + U_m$, where $v_{m+j} = -(m-1) + 0 + 1 + 2 + \cdots + j = -(m-1) + j(j+1)/2$. Note that under the assumption that $m \notin \mathcal{V}$, $v_{m+U_m} < 0$ and $v_{m+U_m} + (U_m + 1) > 1$.

Now, suppose $m + U_m + 1 \leq i \leq 2m - 1$. For $i = U_m + 1$, employing (14) and (4), we have

$$\begin{aligned} y_{m+U_m+1} &\leq A + \frac{A^{v_{m+U_m}}(1 + r_{m+U_m}A)}{A^{-(U_m+1)}(1 - l_{U_m+1}A)} \\ &\leq A + A^{v_{m+U_m}+U_m+1}(1 + r_{m+U_m}A)(1 + (l_{U_m+1} + 1)A) \\ &\leq A + A^2(1 + (r_{m+U_m} + l_{U_m+1} + 2)A) \\ &\leq A + A^2(1 + \delta_1), \end{aligned} \tag{15}$$

for sufficiently small $A > 0$, where $0 < \delta_1 < 1$. Inductively, we have

$$\begin{aligned} y_{m+U_m+i} &\leq A + \frac{A(1 + (1 + \delta_{i-1})A)}{A^{-(U_m+i)}(1 - l_{U_m+i}A)} \\ &\leq A + A^{U_m+i+1}(1 + (\delta_{i-1} + l_{U_m+i} + 2)A) \\ &\leq A + A^2(1 + \delta_i) \end{aligned} \tag{16}$$

for $2 \leq i \leq m - 1 - U_m$, where $0 < \delta_j < 1$, for $1 \leq j \leq m - 1 - U_m$.

Hence suppose $2m \leq i \leq 2m + U_m$. Employing (14) and (16), we have

$$\begin{aligned} y_{2m} &\leq A + \frac{A(1 + (1 + \delta_{q-1})A)}{A^{v_m}(1 - l_m A)} \leq A + A^{1+(m-1)}(1 + (\delta_{q-1} + l_m + 2)A) \\ &\leq A + A^2(1 + \delta_q), \end{aligned} \tag{17}$$

where $q = m - U_m$ and $0 < \delta_q < 1$.

Inductively, we obtain

$$\begin{aligned} y_{2m+i} &\leq A + \frac{A(1 + (1 + \delta_{q+i-1})A)}{A^{v_{m+i}}(1 - l_{m+i}A)} \leq A + A^{(m-1)-i(i+1)/2+1}(1 + (\delta_{q+i-1} + l_{m+i} + 2)A) \\ &\leq A + A^2(1 + \delta_{q+i}), \end{aligned} \tag{18}$$

for $1 \leq i \leq U_m$, where we have used the fact that $(m-1) - i(i+1)/2 + 1 \geq 2$.

Note that $y_i \in (A, A + 2A^2)$ for $m + U_m + 1 \leq i \leq 2m + U_m$, and consider the function $F : (\mathbb{R}^+)^m \to (\mathbb{R}^+)^m$ defined via

$$F((x_1, x_2 \ldots, x_m)) = (x_2, x_3 \ldots, x_m, A + x_m/x_1). \tag{19}$$

Recalling (3), we have shown that the $(2m + U_m + 1)$-th iterate of F, F^{2m+U_m+1}, maps $\mathcal{S}_A = [A, A + 2A^2]^m$ into itself. Since $\mathcal{S}_A$ is homeomorphic to a closed disk of dimension m, and F is continuous, Brouwer's fixed point theorem applies and F^{2m+U_m+1} has a fixed point. Thus, a period-$(2m + U_m + 1)$ solution to (1) exists, as desired.

Note that for sufficiently small A, Eqs. (4) and (14) imply that $y_i \notin (A, A + 2A^2)$ for $0 \leq i \leq m + U_m$, which precludes any smaller period, and hence we have a prime periodic solution of the required form. $\square$

Remark. Note that $\{m > 1 | m \notin \mathcal{V}\} = \{5, 8, 9, 12, 13, 14, 17, 18, 19, \ldots\}$. By tracking constants in the above argument, it is possible, for fixed values of m, to obtain explicit ranges of A for which periodic solutions of the form prescribed by the theorem exist.

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References

[1] R.M. Abu-Saris, R. DeVault, Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Lett. 16 (2) (2003) 173–178.
[2] K.S. Berenhaut, J.D. Foley, S. Stevic, Quantitative bounds for the recursive sequence $y_n = A + \frac{y_{n-1}}{y_{n-k}}$, Appl. Math. Lett. 19 (9) (2006) 983–989.
[3] M. Saleh, M. Aloqeili, On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$, Appl. Math. Comput. 176 (1) (2006) 359–363.
[4] M. Saleh, M. Aloqeili, On the rational difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Comput. 177 (1) (2006) 189–193.
[5] A.M. Amleh, E.A. Grove, G. Ladas, D.A. Georgiou, On the recursive sequence $y_{n+1} = \alpha + \frac{y_{n-1}}{y_n}$, J. Math. Anal. Appl. 233 (1999) 790–798.
[6] K.S. Berenhaut, J.D. Foley, S. Stevic, The global attractivity of the rational difference equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$, Proc. Amer. Math. Soc. 135 (2007) 1133–1140.
[7] K.S. Berenhaut, J.D. Foley, S. Stevic, The global attractivity of the rational difference equation $y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p$, in: Proceedings of the American Mathematical Society, 2006 (in press).
[8] K.S. Berenhaut, S. Stevic, The behaviour of the positive solutions of the difference equation $x_n = A + \left(\frac{x_{n-2}}{x_{n-1}}\right)^p$, J. Difference Equ. Appl. 12 (9) (2006) 909–918.
[9] K.S. Berenhaut, S. Stević, On positive nonoscillatory solutions of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}{}^p}{x_n{}^p}$, J. Difference Equ. Appl. 12 (5) (2006) 495–499.
[10] R. DeVault, C. Kent, W. Kosmala, On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, J. Difference Equ. Appl. 9 (8) (2003) 721–730.
[11] H. El-Metwally, E.A. Grove, G. Ladas, H.D. Voulov, On the global attractivity and the periodic character of some difference equations, J. Difference Equ. Appl. 7 (2001) 837–850.
[12] H.M. El-Owaidy, A.M. Ahmed, M.S. Mousa, On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, J. Appl. Math. Comput. 12 (1–2) (2003) 31–37.
[13] M.R.S. Kulenović, G. Ladas, Dynamics of Second Order Rational Difference Equations, Chapman & Hall/CRC, 2002.
[14] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC Press, Boca Raton, 2004.
[15] V. Kocić, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, in: Mathematics and its Applications, vol. 256, Kluwer Academic Publishers Group, Dordrecht, 1993.
[16] W.T. Patula, H.D. Voulov, On the oscillation and periodic character of a third order rational difference equation, Proc. Amer. Math. Soc. 131 (3) (2003) 905–909.
[17] S. Stević, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}$, J. Appl. Math. Comput. 18 (1–2) (2005) 229–234.